A discussion on the origin of quantum probabilities

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Abstract

We study the origin of quantum probabilities as arising from non-boolean propositional-operational structures. We apply the method developed by Cox to non distributive lattices and deduce non-kolmogorvian probability measures of quantum mechanics. We outline a general framework for deducing probabilities for general propositional lattices generalizing the method developed by Cox.

Key words: quantum probabilities - quantum information - quantum logic - Kolmogorov

1 Introduction

Quantum probabilities have been an intriguing question since the very beginning of quantum theory. It was rapidly realized that probability amplitudes of quantum process obeyed different rules, as for example, the sum rule of probability amplitudes giving rise to interference terms or the nonexistence of joint distributions for noncommuting observables. In 1936 von Neumann wrote the first work ever to introduce quantum logics [1], proving that quantum mechanics requires a propositional calculus substantially different from all classical logics. In the paper, he rigorously isolated a new algebraic structure for quantum logics. The concept of creating a propositional calculus for quantum logic had been already outlined by him in 1932. But in 1936 the necessity of devising a new propositional calculus was demonstrated through several proofs. For example, photons cannot pass through two successive filters which are polarized perpendicularly (e.g. one horizontally and the other vertically), and therefore, a fortiori, it cannot pass if a third filter polarized diagonally is added to the other two, either before or after them in the succession. However, if the third filter is added in-between the other two, the photons do pass through. Such experimental fact is translatable into logic as a non-commutativity of conjunction and lies beyond classical considerations.
Quantum and classical probabilities have points in common as well as differences. These differences has been largely studied in the literature [2, 3, 4, 5, 6, 7, 8, 9].

There exist two important axiomatizations of probabilities. One of them was provided by Kolmogorov [10], a set theoretical approach based on boolean sigma algebras of a sample space. Probabilities are defined as measures over subsets of a given set. Thus, the Kolmogorovian approach is set theoretical and usually identified (but not necessarily) with a frequentistic interpretation of probabilities. It was then realized that quantum probabilities can be formulated as measures over non boolean structures (instead of boolean sigma algebras), therefore the term non-boolean or non-kolmogorovian probabilities [6]. It is remarkable that the creation of quantum theory and the works on the foundations of probability by Kolmogorov where both developed at the same time in the twenties.
An alternative approach to the Kolmogorovian construction of probabilities was developed by R. T. Cox [11, 12]. Cox starts with a propositional calculus, intended to represent assertions which represent our knowledge about the world or system under investigation. As is well known since the work of Boole [13], propositions of classical logic (CL) can be represented as a Boolean lattice, i.e., an algebraic structure endowed with lattice operations "$\wedge$", "$\vee$" and "$\neg$", which are intended to represent conjunction, disjunction and negation respectively, and a partial order relation "$\leq$" which is intended to represent logical implication. Boolean lattices (as seen form an algebraic point of view) can be characterized by axioms [14, 15, 16]. By considering probabilities as an inferential calculus on a boolean lattice, Cox showed that the axioms of probability can be deduced as a consequence of lattice symmetries, as well as entropy as a measure of information. Thus, differently form the set theoretical approach of Kolmogorov, the approach by Cox considers probabilities as an inferential calculus, and because of its natural links with entropy as a measure of information, is directly linked to Jaynes's MaxEnt principle [17, 18].

It was recently shown that Feymann's rules of quantum mechanics can be deduced from boolean structures using a variant of Cox method [19, 20, 21, 22, 23] (see also [14, 15]). This is done by:

- first defining an operational propositional calculus on a quantum system under study (which results a distributive lattice), and after that,
- postulating that any quantum process (interpreted as a proposition in the operational propositional calculus) can be represented by a pair of real numbers and,
- using a variant of the method developed by Cox, showing that these pairs of real numbers obey the sum and product rules of complex numbers, and can then be interpreted as the quantum probability amplitudes which appear in Feymann's rules.

There is a long tradition with regards to the application of lattice theory to physics. The quantum logical (QL) approach to quantum theory (and physics in general), initiated by von Neumann in [24], has been a traditional tool for studies on the foundations of quantum mechanics (see for example [25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 3], and for a complete bibliography [4], [36], and [37]).
The (QL) approach to physics bases itself on defining elementary tests and propositions for quantum systems and then, studying the structure of these propositional structures. This is done in an operational way [27, 38, 39, 40, 41, 42], and is susceptible of considerable generalization to arbitrary physical systems (not necessarily quantum ones). That is why the approach is also called *operational quantum logic* (OQL). One of the most important goals of OQL is to impose operationally motivated axioms on a lattice structure in order that it can be made isomorphic to a projection lattice on a Hilbert space. There are different positions in the literature about the question of whether this goal has been achieved or not, and also, of course, alternative operational approaches to physics, as the *convex operational* one [**?**, **?**, **?**, 2]. The operational approach presented in [2] bases itself only in the convex formulation of any statistical

theory, and it can be shown that the more general structure which appears under reasonable operational considerations is an σ-*orthocomplemented orthomodular poset*, a more general class than *orthomodular lattices* (the ones which appear in quantum theory). We will come back to these issues and review the definitions for these structures below.

In this work we ask the following questions:

- what happens if the Cox's method mentioned above is applied to general lattices (representing general, not necessarily distributive, physical systems)?
- does the logical underlying structure of the theory determine the form and properties of the probabilities?
- is it possible to generalize Cox's method to arbitrary lattices?

As we shall see below, it is possible to use these questions to derive quantum probabilities. We will show that once the operational structure of the theory is determined, the general properties of probability theory are determined. We also discuss the implications of this derivation for the foundations of quantum physics and probability theory, and compare with different approaches: the one presented in [21], the OQL approach, the operational approach of [2], and the traditional one (represented by the von Neumann formalism of Hilbertian quantum mechanics [44]).

The approach presented here shows itself as susceptible of great generalization: we provide an algorithm for developing generalized probabilities using a combination of the Cox's method with the OQL approach. This opens the door to the development of more general probability and information measures. This approach exhibits the advantage that, in the particular case of quantum mechanics, it includes mixed states, unlike other approaches based only on pure states (like the ones presented in [23] and [19]).

The paper is organized as follows. In Section 2 we review the OQL approach to physics as well as lattice theory. In Section 3 we review Kolmogorov's and Cox' approaches to probability. After that, in Section 4, we review quantum probabilities and their differences with classical ones. In Section 5 we discuss the approach developed in [19], [20], [21], [22], and [23]. In Section 6 we apply Cox's method for deriving non-Kolmogorovian probabilities from non-boolean lattices and study several examples. Finally, in section 8 some conclusions are drawn.

2 The lattice/operational approach to physics

The quantum logical approach to physics is vast and includes different programs. We will concentrate on the path followed by von Neumann and the operational approach developed by Jauch, Piron, and others. First, we recall the relationship between projection operators and elementary tests in QM. After studying the examples of lattices applied to QM and CM, as first introduced in [24], we review the main features of the QL approach.

2.1 Elementary notions of lattice theory

A partially ordered set (also called a poset) is a set X endowed with a partial ordering relation "$<$" satisfying

- 1- For all $x, y \in X$, $x < y$ and $y < x$ entails $x = y$
- 2- For all $x, y, z \in X$, if $x < y$ and $y < z$, then $x < z$

The notation "$x \leq y$" is used to denote "$x < y$" or "$x = y$". A lattice $\mathcal{L}$ will be a poset in which any two elements a and b have a unique supremum (the elements' least upper bound "$a \vee b$"; called their join) and an infimum (greatest lower bound "$a \wedge b$"; called their meet). Lattices can also be characterized as algebraic structures satisfying certain axiomatic identities imposed on operations "$\vee$" and "$\wedge$". For a *complete* lattice all its subsets have both a supremum (join) and an infimum (meet).

A *bounded* lattice has a greatest (or maximum) and least (or minimum) element, denoted 1 and 0 by convention (also called top and bottom, respectively). Any lattice can be converted into a bounded lattice by adding a greatest and least element, and every non-empty finite lattice is bounded. For any set A, the collection of all subsets of A (called the power set of A) can be ordered via subset inclusion to obtain a lattice bounded by A itself and the null set. Set intersection and union represent the operations meet and join, respectively.

Every complete lattice is a bounded lattice. While bounded lattice homomorphisms in general preserve only finite joins and meets, complete lattice homomorphisms are required to preserve arbitrary joins and meets. If P is a bounded poset, an orthocomplementation in P is a bounded lattice in which there exists a unary operation "$\neg(\ldots)$" such that:

$$\neg(\neg(a))) = a \tag{1a}$$

$$a \leq b \longrightarrow \neg b \leq \neg a \tag{1b}$$

$a \vee \neg a$ and $a \wedge \neg a$ exist and both

$$a \vee \neg a = \mathbf{1} \tag{1c}$$

$$a \wedge \neg a = \mathbf{0} \tag{1d}$$

hold. A bounded poset with ortocomplementation will be called an orthoposet. An ortholattice, will be an orthoposet which is also a lattice. For a, $b \in \mathcal{L}$ (an ortholattice or orthoposet), we say that a is orthogonal to b ($a \perp b$) iff $a \leq \neg b$.

Distributive lattices are lattices for which the operations of join and meet are distributed over each other. A complete complemented lattice that is also distributive is a Boolean algebra. For a distributive lattice, the complement of x, when it exists, is unique. The prototypical examples of Boolean algebras are collections of sets for which the lattice operations can be given by set union and intersection, and lattice complementation by set theoretical complementation.
A *modular* lattice is one that satisfies the following self-dual condition (*modular law* or *modular identity*)

$$x \leq b \longrightarrow x \vee (a \wedge b) = (x \vee a) \wedge b \tag{2}$$

Modular lattices arise naturally in algebra and in many other areas of mathematics. For example, the subspaces of a finite dimensional vector space form a modular lattice. Every distributive lattice is modular. In a not necessarily modular lattice, there may still be elements b for which the modular law holds in connection with arbitrary elements a and x ($\leq b$). Such an element is called a modular element. Even more generally, the modular law may hold for a fixed pair (a, b). Such a pair is called a modular pair, and there are various generalizations of modularity related to this notion and to semi-modularity.

An orthomodular lattice will be an ortholattice satisfying the orthomodular law:

$$x \leq b \longrightarrow x \vee (\neg x \wedge b) = b \tag{3}$$

For example, the lattice $\mathcal{L}(H)$ of closed subspaces of a Hilbert space H is orthomodular. $\mathcal{L}(H)$ is modular iff H is finite dimensional. In addition, if we give the set $\mathcal{P}_p(H)$ of (bounded) projection operators on H an ordering structure by defining $P \leq Q$ iff $\mathcal{P}(H) \leq \mathcal{Q}(H)$, then $\mathcal{P}_p(H)$ is lattice isomorphic to $\mathcal{L}(H)$, and hence orthomodular [24].

The concept of lattice's atom is of great physical importance. If $\mathcal{L}$ has a null element 0, then an element x of $\mathcal{L}$ is an *atom* if $0 < x$ and there exists no element y of $\mathcal{L}$ such that $0 < y < x$. One says that $\mathcal{L}$ is:
i) *Atomic*, if for every nonzero element x of $\mathcal{L}$, there exists an atom a of $\mathcal{L}$ such that $a = x$
ii) Atomistic, if every element of $\mathcal{L}$ is a supremum of atoms.

2.2 Elementary measurements and projection operators

In QM, an elementary measurement given by a yes-no experiment (i.e., a test in which we get the answer "yes" or the answer "no"), is represented by a projection operator. If $\mathbb{R}$ is the real line, let $B(\mathbb{R})$ be the family of subsets of $\mathbb{R}$ such that

- 1 - The family is closed under set theoretical complements.
- 2 - The family is closed under denumerable unions.
- 3 - The family includes all open intervals.

The elements of $B(\mathbb{R})$ are the *Borel subsets* of $\mathbb{R}$ [45]. In QM, a projection valued measure (PVM) M, is a mapping

$$M : B(\mathcal{R}) \to \mathcal{P}(\mathcal{H}) \tag{4a}$$

such that

$$M(0) = 0 \tag{4b}$$

$$M(\mathcal{R}) = \mathbf{1} \tag{4c}$$

$$M(\cup_j(B_j)) = \sum_j M(B_j), \tag{4d}$$

for any disjoint denumerable family B_j. Also,

$$M(B^c) = \mathbf{1} - M(B) = (M(B))^{\perp} \tag{4e}$$

All operators representing observables can be expressed in terms of PVM's (and so, reduced to sets of elementary measurements), via the spectral decomposition theorem, which asserts that the set of spectral measurements may be put in a bijective correspondence with the set $\mathcal{A}$ of self adjoint operators of $\mathcal{H}$ [45].

Any quantum system represented by a separable Hilbert space $\mathcal{H}$ has associated a lattice, formed by all its closed subspaces $\mathcal{L}_{v\mathcal{N}}(\mathcal{H}) =< \mathcal{P}(\mathcal{H}),\ \cap,\ \oplus,\ \neg,\ 0,\ 1 >$, where 0 is the empty set $\emptyset$, 1 is the total space $\mathcal{H}$, $\oplus$ the closure of the sum, and $\neg(S)$ is the orthogonal complement of a subspace S [5]. Closed subspaces can be put in one to one correspondence with projection operators. *Thus, elementary tests in QM, which are represented by projection operators, can be endowed with a lattice structure.* This lattice was called "Quantum Logic" by Birkhoff and von Neumann [24]. We will refer to this lattice as the *von Neumann-lattice ($\mathcal{L}_{v\mathcal{N}}(\mathcal{H})$)* [5].

The analogous of this structure in Classical Mechanics (CM) was provided by Birkoff and von Neumann [24]. Take for example the following operational propositions on a classical harmonic oscillator: "the energy is equal to E_0" and "the energy is lesser or equal than E_0". The first one corresponds to an ellipse in phase space, and the second to the ellipse and its interior. This simple example shows that operational propositions in CM can be represented by subsets of the phase space. Thus, given a classical system S with phase space Γ, let $\mathcal{P}(\Gamma)$ represent the set formed by all the subsets of Γ. Then, the set of operational propositions can be represented by the elements of $\mathcal{P}(\Gamma)$. Thus, $\mathcal{P}(\Gamma)$, as well as $\mathcal{P}(\mathcal{H})$, can be structured as a lattice of propositions. As we have seen in Section 2.2, the operational propositions of a classical systems S can be identified with the set of subsets $\mathcal{P}(\Gamma)$ of its phase space Γ. This set can be enbdowed with a lattice structure as follows. If "$\vee$" is represented by set union, "$\wedge$" by set intersection, "$\neg$" by set complement (with respect to Γ), and $\mathbf{0}$ and $\mathbf{1}$ are represented by $\emptyset$ and Γ respectively, then $< \mathcal{P}(\Gamma),\ \cap,\ \oplus,\ \neg,\ 0,\ 1 >$ forms a complete bounded lattice. This is the lattice of propositions of a classical system, which as is well known, *is a boolean one*.

2.3 The Quantum Logical Approach to Physics

We have seen that operational propositions of quantum and classical systems can be endowed with lattice structures. These lattices where boolean for classical systems, and non boolean for quantum ones. This fact, discovered by von Neumann [24], raised a lot of interesting questions. The first one is: is it possible to obtain QM (as well as CM) by imposing suitable axioms on a lattice structure? The surprising answer is *yes, it is possible*. But the road which led to this result was fairly difficult and full of obstacles. In the first place, it was a very difficult mathematical task to demonstrate that a suitably chosen set of axioms on a lattice would yield a representation theorem which would allow one to recover Hilbertian QM. The first result was obtained by Piron, and the final demonstration was given by Solèr in 1995 [46] (see also [4], page 72). One of the main advantages of this approach is that the axioms imposed on a lattice structure could be given a clear operational interpretation: unlike the Hilbert space formulation, whose axioms have the disadvantage of being *ad hoc* and physically unmotivated, the quantum logical approach is clearer and more intuitive from a physical point of view.

The second important question raised by the von Neumann discovery was: given that QM and CM can be described by operational lattices, is it possible to formulate the entire apparatus of physics in lattice theory terms? Given *any* physical system, quantum, classical, or obeying more general tenets, it is always possible to define an operational propositional structure on it using the notion of elementary tests. A very general approach to physics can be given using *event structures*, which are sets of events endowed with a probability measures satisfying certain axioms [2]. It can be shown (see [2], Chapter 3) that any event structure is isomorphic to a *σ-orthocomplete orthomodular poset*, which is an orthocomplemented poset $\mathcal{P}$, satisfying the orthomodular identity (3), and for which if $a_i \in \mathcal{P}$ and $a_i \perp a_j$ $(i \neq j)$, implies that $\bigvee a_i$ exists for $i = 1, 2, \ldots$. Remark that event structures (or σ-orthocomplete orthomodular posets) need not to be lattices. However, lattices are very general structures and encompass most important examples. Consequently, we will work with orthomodular lattices in this paper (and indicate which results can be easily extended to σ-orthocomplete orthomodular posets).
There are other general approaches to statistical theories, one of them is the convex operational one [47, 48, 49, 50], which consists on imposing axioms on a convex structure (formed by physical states). Indeed, the convex operational approach is even more general than the quantum logical, but we will not discuss in detail this issue here.

3 Cox vs. Kolmogorov

In this Section we will review two different approaches to probability theory. On one hand, the Cox's approach, in which probabilities are considered as measures of the plausibility of a given event or happenings. On the other hand, the traditional Kolmogorovian one, a set theoretical approach which is compatible with the interpretation of probabilities as frequencies.

3.1 Kolmogorov

Given a set Ω, let us consider a σ-algebra Σ of Ω. Then, a probability measure will be given by a function μ such that

$$\mu : \Sigma \to [0,1] \tag{5a}$$

which satisfies

$$\mu(\emptyset) = 0 \tag{5b}$$

$$\mu(A^c) = 1 - \mu(A), \tag{5c}$$

where $(\ldots)^c$ means set-theoretical-complement and for any pairwise disjoint denumerable family $\{A_i\}_{i\in I}$

$$\mu(\bigcup_{i\in I} A_i) = \sum_i \mu(A_i) \tag{5d}$$

where conditions (5) are the well known axioms of Kolmogorov. The triad (Ω, Σ, μ) is called a *probability space*. Depending on the context, probability spaces obeying Eqs. (5) are usually referred as Kolmogorovian, classical, commutative or boolean probabilities [2].
It is possible to show that if (Ω, Σ, μ) is a kolmogorovian probability space, the *inclusion-exclusion principle* holds

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B) \tag{6}$$

As remarked in [51], Eq. (6) was considered as crucial by von Neumann for the interpretation of $\mu(A)$ and $\mu(B)$ as relative frequencies. If $N_{\mu(A\cup B)}$, $N_{\mu(A)}$, $N_{\mu(B)}$, $N_{\mu(A\cap B)}$ are the number of times of each event to occur in a series of N repetitions, then (6) trivially holds.
This principle does no longer hold in QM, and this is linked to the non-boolean character of this theory. Thus, the relative frequencies interpretation of quantum probabilities is problematic here.

3.2 Cox's approach

Propositions of classical logic can be endowed with a Boolean lattice structure. The logical implication "$\longrightarrow$" is associated with a partial order relation "$\leq$", the conjunction "and" with the greatest lower bound "$\wedge$", disjunction "or" with the lowest upper bound "$\vee$", and negation "not" is associated with complement "$\neg$". A boolean lattice is a structure satisfying the following axioms:

- L1. $x \vee x = x$, $x \wedge x = x$ (idempotence)
- L2. $x \vee y = y \vee x$, $x \wedge y = y \wedge x$ (commutativity)
- L3. $x \vee (y \vee z) = (x \vee y) \vee z$, $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ (associativity)

- L4. $x \vee (x \wedge y) = x \wedge (x \vee y) = x$ (absortion)
- D1. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (distributivity 1)
- D2. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ (distributivity 2)

As is well known, boolean lattices can be represented as subsets of a given set, with "$\leq$" represented as set theoretical inclusion $\subseteq$, "$\vee$" represented as set theoretical union "$\cup$", "$\wedge$" represented as set intersection "$\cap$", $\neg$ represented as the set theoretical complement "$(\ldots)^c$".

As a typical feature, Cox derives classical probability theory as an inferential calculus on boolean lattices. A real valued function φ representing the degree to which a proposition y implies another proposition x is postulated, and its properties deduced from the algebraic properties of the boolean lattice. It turns out that $\varphi(x|y)$ –if suitably normalized– satisfies all the properties of a Kolmogorovian probability (Eqs. (5)). The deduction will be omitted here, and the reader is referred to [12, 11, 14, 15, 20] for detailed expositions.
Despite their formal equivalence, there is a great conceptual difference between the approaches of Kolmogorov and Cox. In the Kolmogorovian approach probabilities are naturally interpreted as relative frequencies in a sample space. On the other hand, the approach developed by Cox, considers probabilities as a measure of the degree of belief of an intelligent agent (which may be a machine), on the truth of proposition x if it is known that y is true. This measure is given by the real number $\varphi(x|y)$, and in this way the Cox's approach is more compatible with a *Bayesian* interpretation of probability theory.

4 Quantum vs. classical probabilities

In this Section we will introduce quantum probabilities and look at their differences with classical ones. Great part of the hardship faced by Birkhoff and von Neumann in developing the logic of quantum mechanics were due to the inadequacies of classical probability theory. Their point of view was that a statistical physical theory could be regarded as a probability theory, founded on a calculus of events. These events should be the experimentally verifiable propositions of the theory, and the structure of this calculus was to be deduced from empirical considerations, which for the quantum case, resulted in an orthomodular lattice [24, 9].
In the formulation of both classical and quantum probabilities, states can be regarded as representing consistent probability assignments [47, 7]. In the quantum mechanics instance *this "states as mappings" visualization* is achieved via *postulating* a function [5]

$$s : \mathcal{P}(\mathcal{H}) \to [0;1] \tag{7a}$$

such that:

$$s(\mathbf{0}) = 0 \ (\mathbf{0} \text{ is the null subspace}). \tag{7b}$$

$$s(P^{\perp}) = 1 - s(P), \tag{7c}$$

and, for a denumerable and pairwise orthogonal family of projections

$$P_j, \ s(\sum_j P_j) = \sum_j s(P_j). \tag{7d}$$

Gleason's theorem [52, 53], tell us that if the dimension of $\mathcal{H} \geq 3$, any measure s satisfying (7) can be put in correspondence with a trace class operator (of trace one) ρ_s via the correspondence:

$$s(P) := \text{tr}(\rho_s P) \tag{8}$$

And vice versa: using equation (8) any trace class operator of trace one defines a measure as in (7). Thus, equations (7) define a probability: to any elementary test (or event), represented by a projection operator P, $s(P)$ gives us the probability that the event P occurs, and this is experimentally granted by the validity of Born's rule. But in fact, (7) is not a classical probability, because it does not obeys Kolmogorov's axioms (5). The main difference comes from the fact that the σ-algebra in (5) is boolean, while $\mathcal{P}(\mathcal{H})$ is not. Thus, quantum probabilities are also called non-kolmogorovian (or non-boolean) probability measures. The crucial fact is that, in the quantum case, *we do not have a σ-algebra, but an orthomodular lattice of projections.*
One of the most important ensuing differences expresses itself in the fact that Eq. (6) is no longer valid in QM. Indeed, it may happen that

$$s(A) + s(B) \leq s(A \vee B) \tag{9}$$

for A and B suitably chosen elementary sharp tests. Another important difference comes from the difficulties which appear when one tries to define a quantum conditional probability (see for example [2] and [6] for a comparison between classical and quantum probabilities). Quantum probabilities may also be considered as a generalization of classical probability theory: while in an arbitrary statistical theory a state will be a normalized measure over a suitable C^*-algebra, the classical case is recovered when the algebra is *commutative* [2].
We are thus faced with the following fact: on the one hand, there exists a generalization of classical probability theory to non-boolean operational structures. On the other hand, Cox derives classical probabilities from the algebraic properties of classical logic. As we shall see in detail below, this readily implies that probabilities in CM are determined by the operational structure of classical propositions (given by subsets of phase space). The question is: is it possible to generalize Cox's method to arbitrary propositional structures (representing the operational propositions of an arbitrary theory) even when they are not boolean? What would we expect to find? We will see that the answer to the first question is *yes*, and for the second, it is reasonable to recover quantum probabilities (Eq. (7)). This approach may serve as a solution for a problem posed by von Neumann. In his words:

> "In order to have probability all you need is a concept of all angles, I mean, other than 90. Now it is perfectly quite true that in geometry, as soon as you can define the right angle, you can define all angles. Another way to put it is that if you take the case of an orthogonal space, those mappings of this space on itself, which leave orthogonality intact, lives all angles intact, in other words, in those systems which can be used as models of the logical background for quantum theory, it is true that as soon as all the ordinary concepts of logic are fixed under some isomorphic transformation, all of probability theory is already fixed... This means however, that one has a formal mechanism in which, logics and probability theory arise simultaneously and are derived simultaneously.[51]"

and, as remarked by M. Redei [51]:

> "It was simultaneous emergence and mutual determination of probability and logic what von Neumann found intriguing and not at all well understood. He very much wanted to have a detailed axiomatic study of this phenomenon because he hoped that it would shed "... a great deal of new light on logics and probability alter the whole formal structure of logics considerably, if one succeeds in deriving this system from first principles, in other words from a suitable set of axioms."(quote) He emphasized

–and this was his last thought in his address– that it was an entirely open problem whether/how such an axiomatic derivation can be carried out."

The problem posed above has remained thus far unanswered, and this work may be considered as concrete step towards its solution. Before entering the subject, let us first review an alternative approach.

5 Alternative derivation of Feymann's rules

Refs. [21], [22], and [23] present a novel derivation of Feyman's rules for quantum mechanics, based on a modern reformulation [20] of Cox's ideas on the foundations of probability [11, 12], as follows: First, an experimental logic of processes is defined for quantum systems. This is done in such a way that the resulting algebra is a boolean one. Given a sequence of n measurements M_1,…,M_n on a given system, with results m_1, m_2, …, m_n, the later are organized in the proposition $A = [m_1, m_2, \ldots, m_n]$, as a particular process.
If each of the m_i's has two possible values, 1 and 2, a possible proposition of three measurements is for example $A_1 = [1, 2, 1]$. Another one could be $A_2 = [1, 1, 2]$ and so on. If we want to "coarse grain" a certain measurement, say M_2, and forget about its particular result, we can unite the two outcomes in a joint outcome $(1, 2)$, yielding the experiment (measurement) $\widetilde{M}_2$. Thus, a possible sequence obtained by the replacement of M_2 by $\widetilde{M}_2$ could be $[1, (1, 2), 1]$. This is used to define a logical operation

$$[m_1, \ldots, (m_i, m_i'), \ldots, m_n] = [m_1, \ldots, m_i, \ldots, m_n] \vee [m_1, \ldots, m_i', \ldots, m_n] \tag{10}$$

It is intended that sequences of measurements can be compounded. For example, if we have $[m_1, m_2]$ and $[m_2, m_3]$, we have also the sequence $[m_1, m_2, m_3]$, paving the way for the general definition

$$[m_1, \ldots, m_j, \ldots, m_n] = [m_1, \ldots, m_j] \cdot [m_j, \ldots, m_n] \tag{11}$$

Thus defined, these operations satisfy

$$A \vee B = B \vee A \tag{12a}$$

$$(A \vee B) \vee C = A \vee (B \vee C) \tag{12b}$$

$$(A \cdot B) \cdot C = A \cdot (B \cdot C) \tag{12c}$$

$$(A \vee B) \cdot C = (A \cdot C) \vee (B \cdot C) \tag{12d}$$

$$C \cdot (A \vee B) = (C \cdot A) \vee (C \cdot B), \tag{12e}$$

yielding a boolean algebra. Remark that the logic derived above is empirical (operational). Thus, its main features depend on contingent facts which have to do with i) the characteristics of the system under study and ii) the definitions made by the experimenter. Accordingly, the fact that the algebra is boolean does not constitute a necessary fact. This should be clear if we take into account that the OQL approach gives rise to a non-boolean lattice by defining different questions and elementary tests. In this sense, both approaches, the one presented in [21] and OQL, deviate from the original work of Cox, because the boolean algebra used by Cox is not

intended to be an empirical one, but the logic that we use as thinking creatures. This is an a priori condition, not a matter to be settled either operationally or empirically.
We had already seen in Section 3 that the method of Cox consists of deriving probability and entropy from the symmetries of a boolean lattice, intended to represent our propositions about the world, while probability is interpreted as a measure of knowledge about an inference calculus. Once equations 12 are written, the set up for the derivation of Feynman's rules is ready. The path to follow now is to apply Cox's method to the symmetries defined by equations 12. But this cannot be done straightforwardly. In order to proceed, an important assumption has to be made: each proposition will be represented by a pair of real numbers. This is a strong *ontological* assumption, justified in [21] using Bohr's complementarity principle. These kind of ontological assumptions are susceptible to the same criticism launched against the OQL approach of Jauch and Piron, i.e., the fact that they are not fully operational. As we shall se below, the method proposed in this article is more direct and systematic, and makes the introduction of these indirect and involved assumptions somewhat clearer.
Once that a pair of real numbers is assigned to any proposition, the authors of [21] reasonably assume that equations 12 induce operations in pairs of reals numbers. If propositions A, B, etc. are represented by pairs of real numbers $\mathbf{a}$, $\mathbf{b}$, etc., then, we should have

$$\mathbf{a} \vee \mathbf{b} = \mathbf{b} \vee \mathbf{a} \tag{13a}$$

$$(\mathbf{a} \vee \mathbf{b}) \vee \mathbf{c} = \mathbf{a} \vee (\mathbf{b} \vee \mathbf{c}) \tag{13b}$$

$$(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c}) \tag{13c}$$

$$(\mathbf{a} \vee \mathbf{b}) \cdot \mathbf{c} = (\mathbf{a} \cdot \mathbf{c}) \vee (\mathbf{b} \cdot \mathbf{c}) \tag{13d}$$

$$\mathbf{c} \cdot (\mathbf{a} \vee \mathbf{b}) = (\mathbf{c} \cdot \mathbf{a}) \vee (\mathbf{c} \cdot \mathbf{b}) \tag{13e}$$

We easily recognize in (13) operations satisfied by the complex numbers' field. If they constituted the only possible instance, propositions represented by pairs of real numbers would be complex numbers, and thus, we could easily have Feyman's rules. However, complex numbers are not the only entities that satisfy 13. There are other, and thus, extra assumptions have to be made in order to restrict possibilities. These additional assumptions are presented in [22] and [21], and improved in [23]. We list them below.

- Pair symmetry
- Additivity condition
- Symmetric bias condition

Leaving aside the fact that these extra assumptions are more or less reasonable (justifications for their use are given in [23]), it is clear that the derivation is quite indirect and artificial: the "experimental" logic is thus defined in order to yield algebraic rules compatible with complex multiplication (and the rest of the strategy is to make further assumptions in order to discard other fields different than complex numbers). Further, the experimental logic characterized by equations 12 is not the only possibility, as we have seen in Section 2.

In the rest of this work, we will apply Cox's method to general propositional structures according to the quantum logical approach. We will see that this allows for a new perspective (including the one presented in [21], [20], and [22]) which sheds light onto the structure of non-boolean probability, and is at the same time susceptible of great generalization, living the door open to the derivation of alternative kinds of probabilities.

Yet another important remark is in order. As noted in the Introduction, the work presented in [21], [20], and [22] -as well as ours- is a combination of two approaches: the one which defines propositions in an empirical way (something which it shares with the OQL approach) and that of Cox. Cox's spirit was to derive probability out of Chomsky's generative propositional structures ingrained in our brain [54], and this boolean structure is independent of any experimental information. This does not imply, though, the empirical logics needs to satisfy the same algebra than pervades our thinking, and that is indeed what happens. In this sense, any derivation involving empirical or operational logics deviates from the original intent of Cox. As we shall see, this is not a problem, but an important advantage.

6 Cox's method applied to non-boolean algebras

As seen in Section 2, *operationally motivated axioms imposed on a lattice's propositional structure can be used to describe quantum mechanics.* Disregarding the discussion about the operational validity of this construction, we are only interested in the fact that the embodiment is feasible. Similar constructions can be made for many physical systems, beyond quantum mechanics: the connection between the theory and experience is given by an event structure (elementary tests), and these events can be organized in a lattice structure in many examples of interest.

Thus, our point of departure will be the fact that physical systems can be represented by propositional lattices, and that these lattices need not be necessarily boolean. We will consider atomic orthomodular lattices. Given a system S, and its propositional lattice $\mathcal{L}$, we proceed to apply Cox's method in order to derive an inferential calculus on $\mathcal{L}$.

6.1 Classical Mechanics

We start with classical mechanics (that theory satisfying Hamilton's equations). Given a classical system S_C, the propositional structure is a boolean one, isomorphic to a perfectly boolean lattice used in our logical language (i.e., it is the one used by Cox). Accordingly, as shown in Section 3, the corresponding probability calculus has to be the one which obeys the laws of Kolmogorov (we will find that it satisfies -in particular- equations 5), and the corresponding information measure is Shanon's, as expected.

6.2 Quantum case

As shown by Birkoff and von Neumann in [24], if we follow the above path and try to define the propositional structure for a quantum system S_Q we find an orthomodular lattice $\mathcal{L}_{v\mathcal{N}}(\mathcal{H})$ isomorphic to the lattice of projections $\mathcal{P}(\mathcal{H})$. What are we going to find if we apply instead Cox's method? It stands to reason that we would encounter a non-boolean probability measure with the properties *postulated* in Section 4 (Eqns. (7)). Let us see that this is indeed the case.

The first thing to remark is that in this derivation we assume that we have a non-boolean lattice $\mathcal{L}_{v\mathcal{N}}(\mathcal{H})$, isomorphic to the lattice of projections $\mathcal{P}(\mathcal{H})$, and we must show that the "degree of implication" measure $s(\cdots)$ demanded by Cox's method must satisfy Eqns. (7). We will only consider the case of prior probabilities, and this will mean that we ask for the probability that a certain event happens, given a state of affairs, i.e., a concrete preparation of the system under certain circumstances (which could be naturally or artificially produced). Thus, we are looking

for a function to the real numbers s such that it is non-negative and $s(P) \leq s(Q)$ whenever $P \leq Q$.
Under these assumptions, let us consider the operation "$\vee$". As the direct sum of subspaces is associative, "$\vee$" will be associative too. If P and Q are orthogonal projections ($P \perp Q$), then, we will have that $P \wedge Q = \mathbf{0}$ (otherwise, there would be a vector in P which is not orthogonal to every vector of Q). Next, we consider the relationship between $s(P)$, $s(Q)$, and $s(P \vee Q)$. As $P \wedge Q = \mathbf{0}$, it should happen that

$$s(P \vee Q) = F(s(P), s(Q)), \tag{14}$$

with F a function to be determined. At this point, it should be clear how to proceed. Add now a third proposition R (notice that, for doing this, we need a space of dimension $d \geq 3$, an interesting analogy with Gleason's theorem), such that $P \perp R$, $Q \perp R$, and $Q \perp P$ (and thus $P \wedge R = \mathbf{0}$, $Q \wedge R = \mathbf{0}$, and $Q \wedge P = \mathbf{0}$). Build now the element $(P \vee Q) \vee R$. Then, because of this symmetry, we arrive to the following result

$$s((P \vee Q) \vee R) = s(P \vee (Q \vee R)), \tag{15}$$

and thus,

$$F(F(s(P), s(Q)), s(R)) = F(s(P), F(s(Q), s(R))). \tag{16}$$

The algebraic properties of associativity for $\vee$ and $\perp$ are the only prerequisite for this result. Thus, proceeding as in [14, 15, 20] (and using the solutions to functional equations studied in [55]), we have that –up to a re-scaling:

$$s((P \vee Q)) = s(P) + s(Q)). \tag{17}$$

For any finite family of projections P_j, $1 \leq j \leq n$, we have $s(P_1 \vee P_2 \vee \cdots \vee P_n) = s(P_1) + s(P_2) + \cdots + s(P_n)$. Now, as any projection P satisfies $P \leq \mathbf{1}$, then $s(P) \leq s(\mathbf{1})$, and we can assume without loss of generality the normalization condition $s(\mathbf{1}) = 1$. Thus, for any denumerable pairwise orthogonal infinite family of projections P_j, we have for each n

$$s(\bigvee_{j=1}^{n} P_j) = \sum_{j=1}^{n} s(P_j) \leq 1. \tag{18}$$

As $s(P_j) \geq 0$ for each j, the sequence $s_n = s(\bigvee_{j=1}^{n} P_j)$ is monotone, bounded from above, and thus converges. We write then

$$s(\bigvee_{j=1}^{\infty} P_j) = \sum_{j=1}^{\infty} s(P_j), \tag{19}$$

and we recover condition (7d) of the axioms of quantum probability. Now, given any proposition $\mathcal{L}_{v\mathcal{N}}(\mathcal{H})$, consider $P^{\perp}$. As $P \vee P^{\perp} = \mathbf{1}$, and P is orthogonal to $P^{\perp}$, we have

$$s(P \vee P^{\perp}) = s(P) + s(P^{\perp}) = s(\mathbf{1}) = 1. \tag{20}$$

In other words

$$s(P^{\perp}) = 1 - s(P), \tag{21}$$

which is nothing but condition (7c). On the other hand, as $\mathbf{0} = \mathbf{0} \vee \mathbf{0}$ and $\mathbf{0} \perp \mathbf{0}$, then $s(\mathbf{0}) = s(\mathbf{0}) + s(\mathbf{0})$, and thus, $s(\mathbf{0}) = 0$, which is condition (7b).
This Section shows that the algebraic properties of $\mathcal{L}_{v\mathcal{N}}$ determine the form of the quantum probabilities, which on the light of this discussion, do not need to be postulated.
Thus, we have proved that s is a probability measure on $\mathcal{L}_{v\mathcal{N}}$. Is there any possibility that s differ from an actual quantum probability (given by a density matrix and the Born's rule)? The answer is no, and this is granted by Gleason's theorem, because we have proved that s satisfies Eqns. (7), and Gleason's theorem leaves no alternative (if the dimension of $\mathcal{H} \geq 3$).
An important question is the following: which will be the effect of non-distributivity? As we saw in Section 4, classical probabilities are sub-additive, i.e., they satisfy

$$\mu(A \cup B) \leq \mu(A) + \mu(B), \tag{22}$$

and this is linked to the stronger assertion of Eq. (6) (see also [5], page 104).
But it is indeed the case that the analogous of Eq. (22) does not hold for quantum probabilities. We show below that this derivation is susceptible of generalization. Indeed, the derivation relies mainly on the algebraic properties of the lattice of projections, i.e., in its *non-boolean lattice* structure.

6.3 A Finite Non-distributive Example

There are many systems of interest which can be represented by finite lattices. Many toy models serve to illustrate special features of different theories. Let us start first by analyzing L_{12}, a non distributive lattice which may be considered as the union of two incompatible experiments [56]. An example of L_{12} is provided by a firefly which flies inside a room. The first experiment is to test if the firefly shines on the right side (r) of the room, or on the left (l), or if it does not shine at all (n). Other experiment consists in testing if the firefly shines at the front of the room (f), or on the back (b) of it, or if it does not shine (n). The Hasse diagram of L_{12} is represented in Fig. 6.3.
Applying the Cox's method to the boolean sublattices of L_{12} (and suitably normalizing), we obtain $P(l) + P(r) + P(f) + P(b) + P(n) = 2 - P(n)$, a quantity which may be greater than 1. We explicitly assert then that the exclusion-inclusion law (6) does not hold. This is due to the global non-distributivity of L_{12}. It is also easy to show that $P(l) + P(r) = P(f) + P(b)$, yielding a non trivial relationship between atoms (something which does not occur in a boolean lattice).

6.4 General Derivation

Let $\mathcal{L}$ be an atomic orthomodular lattice. We will compute prior probabilities, i.e., we will assume that $\mathcal{L}$ represents the propositional structure of a given system –physical or not–, and that we want to ascertain how likely is a given event, represented by a lattice element $a \in \mathcal{L}$, assuming that the system has undergone a preparation, i.e., we assume that there exists a *state of affairs*, represented by the top element $\mathbf{1}$. In other words, we will always assume that the "always true proposition" represents the system in an actual preparation or state of affairs.
We must define a function $s : \mathcal{L} \longrightarrow \mathbb{R}$, such that it is always non-negative

$$s(a) \geq 0 \forall a \in \mathcal{L} \tag{23a}$$

and is also order preserving

$$a \leq b \longrightarrow s(a) \leq s(b). \tag{23b}$$

We will show that under these rather general assumptions, a probability theory can be determined. The order preserving assumption readily implies that $s(a) \leq s(\mathbf{1})$ for all $a \in \mathcal{L}$. We will also assume that $s(\mathbf{1}) = K$, a finite real number.

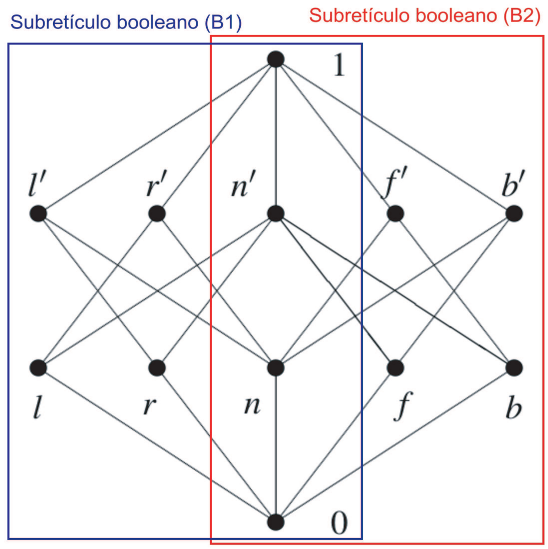

Figure 1: Hasse diagram for L_{12}. The boolean sublattices corresponding to the two complementary contexts are indicated by squared lines.

Now, as an ortholattice is complemented (Eqs. (1)), we will always have that $\neg\neg a = a$ for all $a \in \mathcal{L}$. Accordingly,

$$s(\neg\neg a) = s(a), \tag{24}$$

for all a. Next, it is reasonable to assume that $s(\neg a)$ is a function of $s(\neg a)$, say $s(\neg a) = g(s(a))$. Thus, Eqs. (1a) and (24) imply

$$g(g(s(a))) = s(a), \tag{25}$$

or, in other words,

$$g(g(x)) = x, \tag{26}$$

for positive x. A family of functions which satisfy (26) are $g(x) = x$ and $g(x) = c - x$, where c is a real constant. We discard the first possibility because if true, we would have $s(\mathbf{0}) = s(\neg\mathbf{1}) = g(s(\mathbf{1})) = s(\mathbf{1})$. But if $s(\mathbf{0}) = s(\mathbf{1})$, because of $\mathbf{0} \leq x \leq \mathbf{1}$ for all $x \in \mathcal{L}$, we have $s(\mathbf{0}) = s(x) = s(\mathbf{1})$, and our measure would be trivial. Thus, the only non-trivial option is $s(\neg a) = c - s(a)$ and $s(\mathbf{1}) = k$.
Now, let us see what happens with the "$\vee$" operation. As $\mathcal{L}$ is orthocomplemented, the orthogonality notion for elements is available (see Section 2.1). If $a, b \in \mathcal{L}$ and $a \perp b$, because of (1b), we have that $a \wedge b = \mathbf{0}$. Thud, it is reasonable to assume that $s(a \vee b)$ is a function of $s(a)$ and $s(b)$ only, i.e., $s(a \vee b) = f(s(a), s(b))$. By associativity of the "$\vee$", $(a \vee b) \vee c = a \vee (b \vee c)$ for any $a, b, c \in \mathcal{L}$, and this implies then that $s((a \vee b) \vee c) = s(a \vee (b \vee c))$. If a, b, and c are orthogonal, we will have for the left hand side $s((a \vee b) \vee c) = f(f(s(a), s(b)), s(c))$ and $s(a \vee (b \vee c)) = f(s(a), f(s(b), s(c)))$ for the right hand side. Thus,

$$f(f(s(a),s(b)),s(c)) = f(s(a), f(s(b),s(c))), \tag{27}$$

or, in other words

$$f(f(x,y),z) = f(x, f(y,z)), \tag{28}$$

for all x, y, and z between 0 and 1. As shown in [55], the only solution (up to re-scaling) of (28) is $f(x,y) = x + y$. We have thus shown that if $a \perp b$

$$s(a \vee b) = s(a) + s(b), \tag{29}$$

and we will also have

$$s(a_1 \vee a_2 \cdots \vee a_n) = s(a_1) + s(b_2) + \cdots + s(a_n), \tag{30}$$

whenever a_1, a_2, $\cdots$, a_n are pairwise orthogonal. Suppose now that $\{a_i\}_{i\in\mathbb{N}}$ is a family of pairwise orthogonal elements of $\mathcal{L}$. For any finite n, we have that $a_1 \vee a_2 \vee \cdots \vee a_n \leq \mathbf{1}$, and thus $s(a_1 \vee a_2 \vee \cdots \vee a_n \leq \mathbf{1}) = s(a_1) + s(b_2) + \cdots + s(a_n) \leq s(\mathbf{1}) = K$. Then, $s_n = s(a_1 \vee a_2 \vee \cdots \vee a_n)$ is a monotone sequence bounded from above, and thus it converges to a real number. As $\bigvee\{a_i\}_{i\in\mathbb{N}} = \lim_{n\longrightarrow\infty} \bigvee_{i=1}^{n} a_i$, we can write

$$s(\bigvee\{a_i\}_{i\in\mathbb{N}}) = \sum_{i=1}^{\infty} s(a_i). \tag{31}$$

In any orthomodular lattice we have $\mathbf{1}\perp\mathbf{0}$ (because $\mathbf{0} \leq \neg\mathbf{1} = \mathbf{0}$), and $\mathbf{1} \vee \mathbf{0} = \mathbf{1}$. Thus, $s(\mathbf{1} \vee \mathbf{0}) = s(\mathbf{1}) = s(\mathbf{1}) + s(\mathbf{0})$. Accordingly, $s(\mathbf{0}) = 0$. As $\neg\mathbf{1} = \mathbf{0}$, $s(\mathbf{0}) = c - s(\mathbf{1})$. Thus, $s(\mathbf{1}) = c$ and $c = K$. We will not lose generality if we assume the normalization condition $K = 1$.
The results of this section show that in *any* orthomodular lattice, a reasonable measure s of plausibility of a given event must satisfy

For any orthogonal denumerable family $\{a_i\}_{i\in\mathbb{N}}$

$$s(\bigvee\{a_i\}_{i\in\mathbb{N}}) = \sum_{i=1}^{\infty} s(a_i) \tag{32a}$$

$$s(\neg a) = 1 - s(a) \tag{32b}$$

$$s(\mathbf{0}) = 0. \tag{32c}$$

Why do Eqs. (32) define non-classical (non-Kolmogorovian) probability measures? In a non-distributive orthomodular lattice there always exist elements a and b such that

$$(a \wedge b) \vee (a \wedge \neg b) < a, \tag{33}$$

so that (using $(a \wedge \neg b)\perp(a \wedge b)$, $s((a \wedge \neg b) \vee (a \wedge b)) = s(a \wedge \neg b) + s(a \wedge b) \leq s(a)$. The inequality can be strict, as the quantum case shows. But in any classical probability theory, by virtue of the inclusion-exclusion principle, we always have $s(a \wedge \neg b) + s(a \wedge b) = s(a)$. This simple fact shows that our measures will be non-classical in the general case.

We give in this way an answer to the problem posed by von Neumann and discussed in Section 4: *the algebraic and logical properties of the operational event structure determine uniquely the general form of the probability measures which can be defined over the lattice.* Accordingly, we

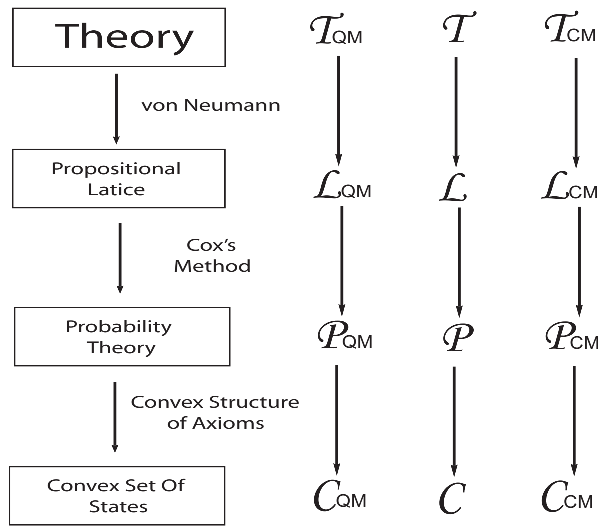

Figure 2: Schematic representation of the method proposed here. A general theory $\mathcal{T}$ determines via the von Neumann approach the algebraic structure of the set of elementary tests. Then, by applying Cox's method, it is possible to determine the general properties of the canonical probability theory assigned to $\mathcal{T}$. Next, by assigning particular values to prior probabilities of atoms, all states will be determined and the form $\mathcal{C}$, the convex set of states. The quantum mechanical (QM) and the classical (CM) case are shown as the extreme instances of a vast family of theories.

did present a generalization of the Cox method to non-boolean structures, namely orthomodular lattices, and then *we have indeed deduced a generalized probability theory.*
Remark that boolean lattices are also orthomodular. This means that our derivation is a generalization of that of Cox, and when we face a boolean structure, classical probability theory will be deduced exactly as in [12, 11]. Another important remark is that the derivation presented here *is also valid* for σ-orthocomplete orthomodular posets (see Section 2 of this work), and thus, of great generality.

7 A general methodology

At this stage it is easy to envisage how a general method can be developed. One starts first by identifying the algebraic structure of the elementary tests of a given theory $\mathcal{T}$. They determine all observable quantities and in the most important examples, they are endowed with a lattice structure. Once the algebraic properties of the pertinent lattice are determined in this way, Cox's method *determines* the general properties of prior probabilities. Of course, it does not determines the particular values that these probabilities may take on the atoms of the theory. The specification of these values amounts to determine a *particular state* of the system under scrutiny. The whole set of these states can be rearranged to yield a convex set. The path we have followed here is illustrated in Fig. 7, with the classical and quantum cases as extreme examples of a vast family.
The method described in this work can be seen as a general epistemological background for a huge family of scientific theories. In order to have a (quantitative) scientific theory, we must be able to make predictions on certain events of interest. Events are regarded here as propositions

susceptible of being tested. Thus, one starts from an inferential calculus which allows for quantifying the degree of belief on the certainty of an event x, if it is known that event y has occurred. The crucial point is that event structures are not always organized as boolean lattices (QM being perhaps the most spectacular example). Thus, in order to determine the general properties of the probabilities of a given theory (and thus all possible states by specifying particular values of prior probabilities on the atoms), we must apply Cox's method to lattices more general than boolean ones.

8 Conclusions

In this work we showed that it is possible to combine Cox's method for the foundations of classical probability theory and the OQL approach, in order to give an alternative derivation of non-kolmogorovian probabilities. In particular, we demonstrated that the usual postulate which states that a measure should be additive for orthogonal projections can be derived using lattice symmetries.
In deriving quantum probability out of the lattice properties, we have shown in a direct way how non-booleanity forbids the derivation of a classical probability. This sheds light on the structure of quantum probabilities and on their differences with the classical case.
Using Cox's approach, Shanon's entropy can be deduced as a natural measure of information over the boolean algebra of classical propositions [12, 23, 20]. We will provide a detailed study of what happens with orthomodular latices elsewhere.
The strategy followed in this work suggests that we are at the gates of a great generalization. The general rule for constructing probabilities (and if we are lucky, entropies) would read as follows:

- 1 - We start by identifying the operational logic of our physical system. The characteristics of this "empirical" logic depends both on physical properties of the system and on the election of the properties that we assume in order to study the system. This can be done in a standard way, and the method is provided by the OQL approach.

- 2 - Once the operational logic is identified, the symmetries of the lattice are used to define the properties of the "degree of implication" function, which will turn out to be the probability function associated to that particular logic. Remark that the same physical system may have different propositional structures, depending of the election of the observers. For example, if we look at the observable "electron's" charge, we will face classical propositions, but if wee look at its momentum and position, we will have a non-boolean lattice.

This method has not only a physical interest, but a mathematical one, namely, solving the problem of characterizing probability measures over general lattices. A final remark: our approach deviates from that of Cox, in the sense that we look for an empirical logic which would be intrinsic to the system under study, and because of that, not only referred to our ignorance about it, but to assumptions about its nature.

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